MTH 1420, SPRING 2012
DR. GRAHAM-SQUIRE

LAB 3: INTEGRALS FOR POWERS OF SINE AND COSINE

Names: ____

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1. Introduction

We have seen already that integration is often more complicated than differentiation, and in the next few lessons we will be discussing different tricks/techniques that you can use to evaluate certain integrals. In this lab we look at taking integrals of sine and cosine functions that are raised to a power. This will lay the groundwork for more elaborate trigonometric substitutions we will do in section 5.7.

2. Instructions

(1) Introduce yourself to your lab partner(s).
(2) Work on the problems together with your partner for the remainder of the lab time. If you are confused about something, talk to your lab partner and explain your question to them to see if they can help. If everyone in the group is stumped, come talk to me for a hint. If you do not finish, it is okay to split up the remaining parts and work on them individually. However, you should meet up sometime outside of class to check each other's work before you turn in a final draft next week.
(3) Your group should write up and turn in one completed lab at the start of the next lab period. You can use this sheet as a cover sheet for the lab you turn in. Each member of the group should write up at least part of the lab, but you should check each other's work since everyone in the group gets the same score.

3. Example

Consider the integral $\int \sin^3 x\, dx$. It is not immediately clear how to find an antiderivative, or what you would choose for u in order to do substitution.

Exercise 1. Try to do the u substitution $u = \sin x$. Does it work? If it does, take the integral and then show that your answer is correct by taking the derivative and getting $\sin^3 x$. If it does not work, explain what the problem is.

You should notice that in order to do an effective u substitution for an integral of this type, you must have both a $\sin x$ and a $\cos x$ in the integrand to make things work out. This is possible, but we have to use some properties of trigonometric functions to do it.

Exercise 2. (a) Because $\sin^2 x + \cos^2 x = 1$, we can subtract the cosine to the other side to get $\sin^2 x =$ ______________.

(b) To integrate $\int \sin^3 x\, dx$, follow these steps: first, rewrite the integrand $\sin^3 x$ to be $(\sin^2 x)(\sin x)$, and substitute what you got from part (a) into the integral. Now try u substitution again with $u = \cos x$ and you should be able to integrate the function. Complete the integration, and then take the derivative of your result to show that it gives you $\sin^3 x$.

4. Method to integrate odd powers of sine and cosine

It turns out that we can do the same trick as what we did in the example above for $\sin^k x$ or $\cos^k x$ for any positive odd number k. The general method is this:

(1) Factor one $\sin x$ or $\cos x$ from the integrand, so that the other factor has an even power.
(2) Use $\sin^2 x + \cos^2 x = 1$ to solve for the appropriate term, and then raise it to whatever even power is appropriate (this becomes more difficult with higher powers).
(3) Substitute in for the even power and do the correct u substitution (that is, let u be the opposite function from what you started with), then integrate.

Exercise 3. Use the method above to calculate $\int \cos^5 x\, dx$.

5. Even powers of sine and cosine

Unfortunately, the trick above does not work when there is an even power for $\sin x$ or $\cos x$. In these situations, you need to use one of the following (half-angle) formulas:

- $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
- $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

Example 4. Evaluate the definite integral $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^2 x\, dx$.

Answer:

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \cos^2 x\, dx &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos(2x)\, dx \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos(2x))\, dx \\
&= \frac{1}{2}(x + \frac{1}{2}\sin(2x))\Big]_{-\pi/2}^{\pi/2} \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise 5. Evaluate the definite integral $\displaystyle\int_{0}^{\pi/2} \sin^4 x\, dx$.